MATH2050C Assignment 4

no. 6cd, 10a, 11a, 14ab, 16cd, 19d, 20, 21.

Section 3.2

(14b) **Solution** Use Squeeze Theorem in $1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}$ and $\lim_{n\to\infty} n^{1/n} = 1$.

(19d) **Solution** Use

$$\frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\frac{3}{n}\frac{4}{n}\cdots\frac{n}{n} \leq \frac{1}{n}\frac{2}{n}\frac{n}{n}\frac{n}{n}\cdots\frac{n}{n} = \frac{2}{n^2}.$$

By Squeeze Theorem we get

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \ .$$

(21) (a) Consider $\{1/n\}$. We have $1/n \to 0$ but $1/n^{1/n} \to 1$.

(b) Simply consider $\{n\}$.

Supplementary Problems

1. Suppose that $x_n \to x, x_n \geq 0$. Show that $x_n^{p/q} \to x^{p/q}$ for $p, q \in \mathbb{N}$.

   Solution Let $x_n \to x$ as $n \to \infty$. Fix some M so thatg $|x_n|, |x| \leq M$. First we claim that for $p \in \mathbb{N}, x_n^p \to x^p$. Indeed,

   $$|x_n^p - x^p| = |(x_n - x)(x_n^{p-1} + x_n^{p-2}x + \cdots + x^{p-1})| \leq pM^{p-1}|x_n - x| \ .$$

   (There are p many terms in the second factor where each term is bounded by M^{p-1}.) By Proposition 3.1 or the Squeeze Theorem we conclude $x_n^p \to x^p$. Next, by a similar argument we can show that $x_n \to x$ implies $x_n^{1/q} \to x^{1/q}$. Now, as $x_n \to x$, $x_n^p \to x^p$. Taking $y_n = x_n^p$ and $y = x^p$, $y_n \to y$ implies $y_n^{1/q} \to y^{1/q}$, that is, $x_n^{p/q} \to x^{p/q}$.

2. Find (a) $\lim_{n\to\infty} n^{1/n^2}$ and (b) $\lim_{n\to\infty} (\frac{n+5}{n-4})^{(n+1)/n}, n \geq 5$.

   Solution (a) Using Squeeze Theorem in $1 \leq n^{1/n^2} \leq n^{1/n}$ to conclude $\lim_{n\to\infty} n^{1/n^2} = 1$.

   (b) $(n+5)/(n-4) > 1$ implies $1 < (\frac{n+5}{n-4})^{(n+1)/n} \leq (\frac{n+5}{n-4})^2$. By the Squeeze Theorem $\lim_{n\to\infty} (\frac{n+5}{n-4})^{(n+1)/n} = 1$.

3. Determine the limit of

   $$\left(1 - \frac{a}{n^2}\right)^n \ , \ a > 0 \ .$$

   Hint: Use Bernoulli's inequality.

   Solution Recall that Bernoulli's inequality $(1+x)^n \geq 1+nx, \ x > -1$. For some large n_0, $-a/n^2 > -1$, and we have $(1-a/n^2)^n \geq 1 - na/n^2 = 1 - a/n$ for all $n \geq n_0$. Therefore, $1 - a/n \leq (1-a/n^2)^n \leq 1, \ n \geq n_0$, and $\lim_{n\to\infty}(1-a/n^2)^n = 1$ by the Squeeze Theorem.

4. Suppose that $\lim_{n\to\infty} x_n = x$. Prove that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x \ .$$

Solution For $\varepsilon > 0$, fix an n_0 such that $|x_n - x| < \varepsilon/3$ for all $n \geq n_0$. Then fix n_1 such that $|(x_1 + \cdots + x_{n_0-1})/n \leq \varepsilon/3$ for all $n \geq n_1$ and some n_2 such that $(n_0 - 1)|x|/n < \varepsilon/3$ for all $n \geq n_2$. Then, for $n \geq \max\{n_0, n_1, n_2\}$,

$$\begin{aligned}
\left|\frac{x_1 + \cdots + x_n}{n} - x\right| &= \left|\frac{x_1 + \cdots + x_{n_0-1}}{n} + \frac{(x_{n_0} - x) + \cdots + (x_n - x)}{n} - \frac{(n_0 - 1)x}{n}\right| \\
&\leq \left|\frac{x_1 + \cdots + x_{n_0-1}}{n}\right| + \left|\frac{(x_{n_0} - x) + \cdots + (x_n - x)}{n}\right| + \left|\frac{(n_0 - 1)|x|}{n}\right| \\
&< \frac{\varepsilon}{3} + \frac{n - n_0 + 1}{n}\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ .
\end{aligned}$$

Remark This problem shows sometimes we need to go back to the definition to prove the limit exists.